Exceptional Precision of a Nonlinear Optical Sensor at a Square-Root Singularity

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Exceptional points (EPs)—spectral singularities of non-Hermitian linear systems—have recently attracted interest for sensing. While initial proposals and experiments focused on enhanced sensitivities neglecting noise, subsequent studies revealed issues with EP sensors in noisy environments. Here we propose a single-mode Kerr-nonlinear resonator for exceptional sensing in noisy environments. Based on the resonator's dynamic hysteresis, we define a signal that displays a square-root singularity reminiscent of an EP. However, our sensor has crucial fundamental and practical advantages over EP sensors: the signal-to-noise ratio increases with the measurement speed, the square-root singularity is easily detected through intensity measurements, and both sensing precision and information content of the signal are enhanced around the singularity. Our sensor also overcomes the fundamental trade-off between precision and averaging time characterizing all linear sensors. All these unconventional features open up new opportunities for fast and precise sensing using hysteretic resonators.

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In 2014, Wiersig proposed using a non-Hermitian degeneracy known as an exceptional point (EP) for sensing [1]. An EP occurs when a pair of eigenvalues and eigenvectors of a non-Hermitian Hamiltonian coalesce. Two coupled linear resonators constitute the typical system where EPs have been observed [2–10] and used for sensing [11–14]. Setting $\hbar = 1$, the coupled resonators are described by a 2×2 Hamiltonian with complex frequencies $\tilde{\omega}_j$ ($j = 1, 2$) in the diagonal and coupling constant g in the off diagonal. The Hamiltonian's eigenvalues are

$$\omega_{\pm} = \tilde{\omega}_{\mathrm{av}} \pm \frac{\tilde{\Delta}}{2}\sqrt{1 + \left(\frac{2g}{\tilde{\Delta}}\right)^2}, \tag{1}$$

with $\tilde{\omega}_{\mathrm{av}} = (\tilde{\omega}_1 + \tilde{\omega}_2)/2$ the average complex frequency and $\tilde{\Delta} = \tilde{\omega}_1 - \tilde{\omega}_2$ the complex detuning [15]. Notice the square-root singularity for $2g/\tilde{\Delta} = \pm i$, where the eigenvalues ω_+ and ω_- coalesce; this is the EP. At the EP, a perturbation to a resonance frequency [i.e., $\Re[\tilde{\omega}_j] \to \Re[\tilde{\omega}_j] + \epsilon$ ($j = 1$ or 2)] results in a splitting $\Re[\omega_+ - \omega_-] \propto \sqrt{\epsilon}$. Essentially, Wiersig proposed using this frequency splitting and the associated linewidth splitting $\Im[\omega_+ - \omega_-]$ as signals for sensing. Unlike conventional sensors where signals scale linearly with ϵ [16–22], the $\sqrt{\epsilon}$ scaling near an EP promised greater sensitivities for small ϵ [1].

Wiersig's proposal met great enthusiasm and skepticism recently. On one hand, experimental claims of enhanced sensitivities [11–13] and proposed applications [14,23–31] of EP sensors have generated excitement [32,33]. On the other hand, it has been argued that the precision of EP sensors is degraded by noise [34–36]. The observation of enhanced fluctuations near an EP supports this criticism [37]. The foregoing debate reveals that the sensitivity, i.e., scaling of signal with perturbation, is insufficient to characterize sensing performance. Particularly important are the effects of noise, which ultimately determine the magnitude of the perturbation that can be detected within a certain measurement time.

In this Letter we propose and numerically demonstrate optical sensing beyond the constraints of linear sensors using a single coherently driven Kerr-nonlinear resonator. We propose to measure the splitting in transmitted intensities at the endpoints of a hysteresis cycle. This intensity splitting scales with the square root of the perturbation strength. Remarkably, the sensing precision and information content of the signal are enhanced around the square-root singularity. Our sensor also exhibits a signal-to-noise ratio that increases with the measurement speed, and an anomalous scaling of the precision with the averaging time. Crucial for practical applications, our approach only requires monochromatic intensity measurements, and avoids the cumbersome and error-prone task of fitting spectral line shapes to extract complex eigenvalues (as done in EP sensors).

Our sensor can be realized using Fabry-Pérot [38–47], whispering-gallery mode [17,18,21,22,48], ring [49–51], photonic crystal [52–57], or any cavity architecture where one mode is spectrally distant from all other modes and probes an intensity-dependent refractive index. In a frame rotating at the driving frequency ω, the intracavity field α satisfies

$$i\dot{\alpha}(t) = \left[-\Delta - \frac{i\Gamma}{2} + U(|\alpha(t)|^2 - 1)\right]\alpha(t) + i\sqrt{\kappa_L}F + D\xi(t). \tag{2}$$

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$\Delta = \omega - \omega_0$ is the laser's detuning from the resonance frequency ω_0. The total loss rate $\Gamma = \gamma + \kappa_L + \kappa_R$ includes intrinsic loss at rate γ and leakage through "left" and "right" input-output ports at rates κ_L and κ_R. U is the Kerr nonlinearity strength. $D\xi(t) = D[\xi_1(t) + i\xi_2(t)]/\sqrt{2}$ represents Gaussian white noise with variance D^2 in the field quadratures. $\xi_j(t)$ have zero mean $[\langle \xi_j(t)\rangle = 0]$ and correlation $\langle \xi_j(t)\xi_k(t+t')\rangle = \delta_{j,k}\delta(t')$. The validity of Eq. (2) for $U \ll \Gamma$ (henceforth assumed) has been verified many times, through quantitative agreement with experiments and full quantum calculations [58].

As usual in optical sensing [17,18,21,22,40,42,45,48], our goal is to detect a perturbation ϵ to ω_0. For this purpose, we define the signal in a way that is inspired (but not restricted) by the behavior of the steady-state solutions to Eq. (2) ($\dot{\alpha} = D = 0$) near the onset of bistability. There, two stable states with different intracavity intensity $N = |\alpha|^2$ exist at a single driving condition. Figure 1(a) shows bistability for $\Delta = \Gamma$ and variable $F/\sqrt{\Gamma}$. Solid and dotted gray curves are stable and unstable steady states, respectively. The bistability range is bound by the turning points $\tilde{N}_\pm$ [red circles in Fig. 1(a)], obtained by setting $d|F|^2/dN = 0$ in Eq. (2):

$$\tilde{N}_\pm = \frac{2\Delta}{3U} \pm \frac{2\Delta}{6U}\sqrt{1 - \left(\frac{\sqrt{3}\Gamma}{2\Delta}\right)^2}. \quad (3)$$

Notice the resemblance to Eq. (1): $\tilde{N}_\pm$ are defined by a square-root function with singularity at the critical detuning $\Delta_c = \sqrt{3}\Gamma/2$. $\tilde{N}_\pm$ coalesce at Δ_c, just like $\omega_\pm$ coalesce at the EP. This suggests using $\tilde{N}_+ - \tilde{N}_-$ as a signal for sensing. However, $\tilde{N}_\pm$ are steady-state solutions expected in quasistatic protocols only. Fast protocols display no sharp turns in N thereby making $\tilde{N}_\pm$ ill defined. This is illustrated by the thin black curves in Fig. 1(a), obtained by scanning $F/\sqrt{\Gamma}$ from 0 to 10 and back within a time $T = 10^4/\Gamma$. Clearly, $\tilde{N}_\pm$ cannot be used for fast sensing. We therefore turn our attention to the crossing points $N_\pm$, where upward and downward scans intersect. $N_\pm$ are marked in Fig. 1(a) by purple (black) circles for the dynamic (static) case.

Figure 1(b) compares the crossing points $N_\pm$ to the turning points $\tilde{N}_\pm$ as Δ (and hence ϵ) varies. For adiabatic protocols following the steady-state solutions, $N_\pm$ (black curves) and $\tilde{N}_\pm$ (red curves) both bifurcate at Δ_c. However, $N_\pm$ offer greater sensitivity in adiabatic protocols since $N_+ - N_- \geq \tilde{N}_+ - \tilde{N}_-$. More importantly, $N_\pm$ are well defined and display the desired square-root scaling even for nonadiabatic protocols. At high speeds the square-root singularity lies below Δ_c, where there is no bistability or static hysteresis; see where the purple curve bifurcates in Fig. 1(b). Nonetheless, dynamic hysteresis still emerges [43,59], and the intensity splitting $\delta N = N_+ - N_-$ can be

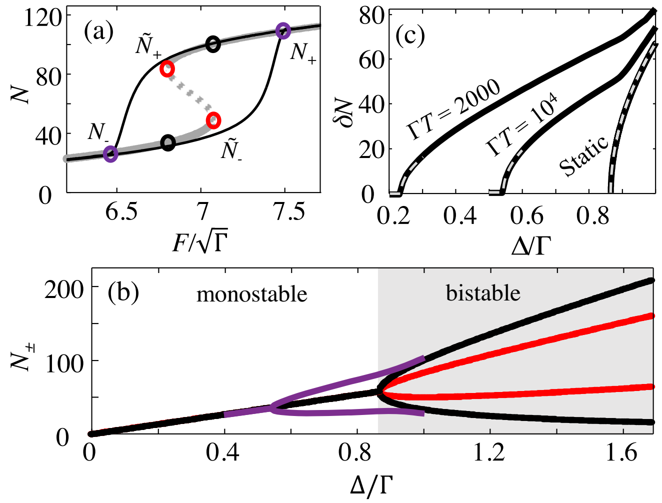

FIG. 1. (a) Intracavity photon number N versus driving amplitude F referenced to the loss rate Γ. The laser-cavity detuning is $\Delta = \Gamma$, and there is no noise. Gray solid and dotted curves represent stable and unstable steady states, respectively. Thin black curves represent the dynamic hysteresis obtained by linearly scanning $F/\sqrt{\Gamma}$ from 0 to 10 and back within a time $T = 10^4/\Gamma$. Red open circles indicate the turning points $\tilde{N}_\pm$. Purple open circles indicate the crossing points $N_\pm$ for the dynamic case, which are used in Figs. 2 and 3 as a signal for sensing. Black open circles indicate $N_\pm$ in the adiabatic limit $\Gamma T \to \infty$. Inset: schematic of the proposed sensor, i.e., a Kerr-nonlinear resonator. (b) Red curves are the turning points $\tilde{N}_\pm$, and black curves are the crossing points $N_\pm$, both in the adiabatic limit. $N_\pm$ in the dynamic case are shown in purple. Parameter values: $\Gamma = 1$, $\gamma = \Gamma/6$, $\kappa_L = \Gamma/2$, $\kappa_R = \Gamma/3$, $U = \Gamma/100$. (c) Solid curves represent the splitting $\delta N = N_+ - N_-$ proposed as a signal for sensing, as function of Δ/Γ. Two curves correspond to scans $F(t)$ within different time T, and the other curve corresponds to the adiabatic limit. Dashed lines are square-root fits as explained in the text.

unambiguously defined as a signal for fast sensing. Practically, δN is determined by measuring the time-dependent intensities N_f and N_b when ramping F forward and backward, respectively. N_+ and N_- are then the first values of N at which $N_b - N_f = 0$ when F increases and decreases, starting from the center of the hysteresis where $N_b - N_f$ is maximum.

Figure 1(c) shows how δN scales with Δ/Γ for two nonadiabatic protocols, one with period $T = 2000/\Gamma$ and another with $T = 10^4/\Gamma$. The static δN, corresponding to $T \to \infty$, is shown for reference. All δN are fitted (see dashed gray curves) with square-root functions near the singularity at Δ_{SS}; this is the point where N_+ and N_- coalesce. The excellent fits evidence that the desired square-root scaling persists for detunings below the static bistability threshold Δ_c and for highly nonadiabatic protocols. Actually, our approach works for any positive detuning, but $\Delta < \Delta_c$ is advantageous for fast sensing because Δ_{SS} decreases with speed as Fig. 1(c) shows.

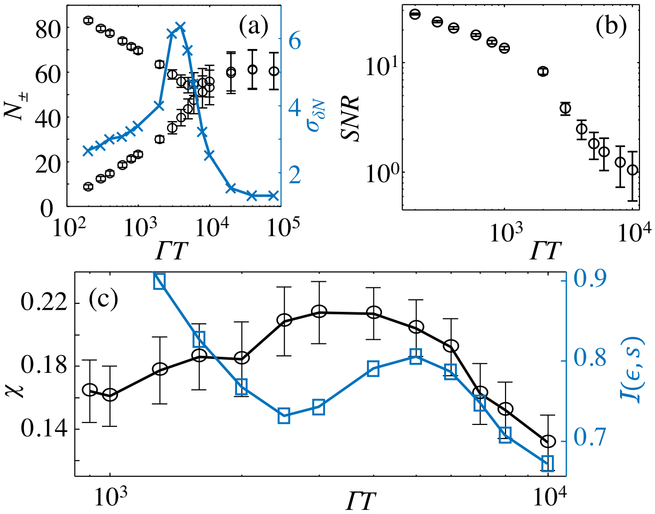

FIG. 2. (a) Black circles are the crossing points $N_\pm$ comprising the signal $\delta N = N_+ - N_-$. Blue crosses are the standard deviation of δN, i.e., $\sigma_{\delta N}$. Both δN and $\sigma_{\delta N}$ are shown for variable ramp time T referenced to Γ. (b) Signal-to-noise ratio $\delta N/\sigma_{\delta N}$ versus ΓT. (c) Precision figure of merit χ versus ΓT. Parameters are as in Fig. 1, with $\Delta/\Gamma = 0.7$ and $D/F_{\mathrm{avg}} = 1/50$. Each point in (a),(b) is calculated based on 1200 individual cycles with different noise realizations. Error bars indicate 1 standard deviation of the mean. Errors in (c) are based on ten calculations of χ, each calculation involving 1200 noise realizations.

However, there is a trade-off between measurement speed and sensitivity: faster protocols decrease the prefactor in the square-root scaling of δN with Δ. In the Supplemental Material we quantify this trade-off and discuss the role of model parameters in general [60].

Next we assess the effects of noise by numerically solving Eq. (2) using the xSPDE MATLAB toolbox [62]. Equation (2) only contains additive noise $\xi(t)$, representing fluctuations in the laser's amplitude and phase and dissipation-induced fluctuations of the intracavity field; the effect of detuning noise is discussed in the Supplemental Material [60]. We consider a single hysteresis cycle of duration T, which also determines the measurement time. Figure 2(a) shows the crossing points $N_\pm$ comprising δN, and the standard deviation $\sigma_{\delta N}$ of δN, both as a function of ΓT. The calculations are done for fixed $\Delta = 0.7\Gamma$ and $D/F_{\mathrm{avg}} = 1/50$, with F_{avg} the average driving amplitude. $\sigma_{\delta N}$ is obtained by calculating δN for 1200 different noise realizations.

Figure 2(a) shows that N_+ and N_- are approximately equal for large ΓT. Indeed, there is no hysteresis in the adiabatic limit for the selected $\Delta = 0.7\Gamma$. Therefore, contrary to conventional sensors, our sensor's performance can be enhanced by reducing the measurement time T. The signal $\delta N = N_+ - N_-$ only becomes appreciable below a critical time T_{SS}, where the system crosses the square-root singularity. This way of approaching a square-root singularity (by varying the ramp time) is advantageous over the usual approach in EP sensors, where the detuning and/or the losses of the resonators [2,5,6,8,10] are slowly varied. Our approach can be orders of magnitude faster thanks to the availability of high-frequency amplitude modulators.

Figure 2(a) also shows how the fluctuations in δN scale with T. The peak in $\sigma_{\delta N}$ at $\Gamma T \approx 4 \times 10^3$ evidences enhanced fluctuations around the square-root singularity. This peak is reminiscent of the enhanced fluctuations at an EP [37], which are at the heart of the aforementioned debate [34–36,63]. While this effect seems discouraging, our statistical analysis below proves that a sensing advantage remains at the square-root singularity.

Figure 2(b) shows the signal-to-noise ratio $\mathrm{SNR} = \delta N/\sigma_{\delta N}$, following a double power law decay with ΓT. The transition between power laws occurs around T_{SS}, where signal fluctuations are enhanced. The SNR follows a similar scaling with ΓT as the dynamic hysteresis area [43,64], resulting in stronger signals at high speeds. However, the laser power needed to cross the full hysteresis range (and hence to measure the signal) also increases with the speed [64]. Hence, there is a trade-off between energy consumption and measurement speed.

Figures 2(a) and 2(b) suggest that, if detection speed is most important and power is available, the cycle time T should be reduced as much as possible and thereby disregard the square-root singularity location. However, for many sensors, precision is also important. A precise measurement is one in which the mean change in the signal due to the perturbation is large compared with the uncertainty in that measurement. In this vein, we define $\chi = (\overline{\delta N}_\epsilon - \overline{\delta N}_0)/(\sigma_0 + \sigma_\epsilon)$ to quantify the precision. $\overline{\delta N}_\epsilon$ and $\overline{\delta N}_0$ are the mean splitting measured for the perturbed and unperturbed cavity, respectively. σ_0 and σ_ϵ are standard deviations corresponding to those signals. Thus, χ quantifies the mean change in the signal relative to the measurement uncertainty.

Figure 2(c) shows χ versus ΓT. For each T, we performed 12×10^3 simulations with different realizations of the noise for a perturbed ($\epsilon = \Gamma/100$) and an unperturbed cavity. We then calculated χ based on the means and standard deviations of the distributions of signals measured for the two cavities. Interestingly, the peak in χ approximately coincides with T_{SS}, thereby demonstrating the precision enhancement by the square-root singularity. While χ remains below 1 (a commonly used detection threshold) in Fig. 2(c), a reliable detection strategy can still be constructed for small χ by allowing a greater probability of missed detection [65]. Overall, Fig. 2(c) reveals a trade-off between measurement time (defined by T) and precision. If precision matters most, one should modulate F with period $T \approx T_{\mathrm{SS}}$. However, if speed is crucial, $T < T_{\mathrm{SS}}$ can be selected.

In the Supplemental Material we show that our nonlinear sensor can compete or outperform a linear sensor in certain parameter regimes [60]. For the comparison, we took equal dissipation, noise strength, detuning, and average driving power. However, a direct comparison is impossible for two reasons mainly. First, our sensor's performance depends on the parameter U, absent in linear sensors. Second, for linear resonators the SNR increases with power, but for our sensor mainly the cycle time T determines the SNR. Despite these differences, our results demonstrate that our sensing strategy can compete with linear sensors. Crucially, our claim does not rely on static sensitivities only, and our approach embraces nonlinearities which typically degrade the precision of linear sensors.

Next we assess our sensor's performance using information theory. We are interested in the mutual information between a perturbation ϵ and its induced signal shift $\mathcal{S} = \delta N_\epsilon - \overline{\delta N_0}$, given by

$$\mathcal{I}(\epsilon;\mathcal{S}) = \sum_{s\in S}\sum_{\varepsilon\in E} p_{(\epsilon,\mathcal{S})}(\varepsilon,s)\log\frac{p_{(\epsilon,\mathcal{S})}(\varepsilon,s)}{p_\epsilon(\varepsilon)p_{\mathcal{S}}(s)}. \quad (4)$$

$p_\epsilon(\varepsilon)$ and $p_{\mathcal{S}}(s)$ are marginal distributions representing our uncertainty in the perturbation and signal, respectively, and $p_{(\epsilon,\mathcal{S})}(\varepsilon,s)$ is their joint probability distribution. $\mathcal{I}(\epsilon;\mathcal{S})$ quantifies the information ϵ and $\mathcal{S}$ share, or how much knowledge of $\mathcal{S}$ reduces uncertainty of ϵ [66].

We model our uncertainty in the perturbation by defining $p_\epsilon(\varepsilon)$ as a Gaussian distribution with mean $\Gamma/100$ and standard deviation $\Gamma/1000$. Then, we determine $p_{\mathcal{S}}(s)$ by numerically calculating $\mathcal{S}$ using 1200 different noise seeds for each cycle time ΓT. This involves calculating δN for the unperturbed (detuning Δ_0) and perturbed ($\Delta_\epsilon = \Delta_0 + \epsilon$) cavity, with $\Delta_0 = 0.7\Gamma$ and ϵ drawn from $p_\epsilon(\varepsilon)$. Finally, we determine $p_{(\epsilon,\mathcal{S})}(\varepsilon,s)$ based on the value of $\mathcal{S}$ for each member of $p_\epsilon(\varepsilon)$.

Figure 2(c) shows that $\mathcal{I}$ is a nonmonotonic function of ΓT. For small ΓT, $\mathcal{I}$ increases with decreasing ΓT; this is expected based on the growing SNR as shown in Fig. 2(b). For large ΓT, $\mathcal{I} \to 0$ as δN becomes increasingly independent of ϵ. Interestingly, for intermediate ΓT, $\mathcal{I}$ peaks around the peak in χ and close to T_{SS}. This demonstrates the correlation between the square-root singularity, $\mathcal{I}$, and the precision. To the best of our knowledge, this is the first demonstration of an information-content enhancement by a square-root singularity, linear or nonlinear. However, the enhancement is only local since much faster scans away from the singularity yield even larger $\mathcal{I}$. Finally, we note that while the exact value of $\mathcal{I}$ depends on the properties of $p_\epsilon(\varepsilon)$, the existence of a peak around T_{SS} (our main result) is independent of those properties as long as $\epsilon \ll \Gamma$ which generally holds in experiments.

Next we assess the effects of averaging. Here, again, our sensor departs from convention. Figures 3(a)–3(c) show typical trajectories of the intensity N obtained by averaging n cycles resulting from an identical protocol $F(t)$ and different noise realizations. The circles in Figs. 3(a)–3(c) indicate the crossing points, whose difference defines δN.

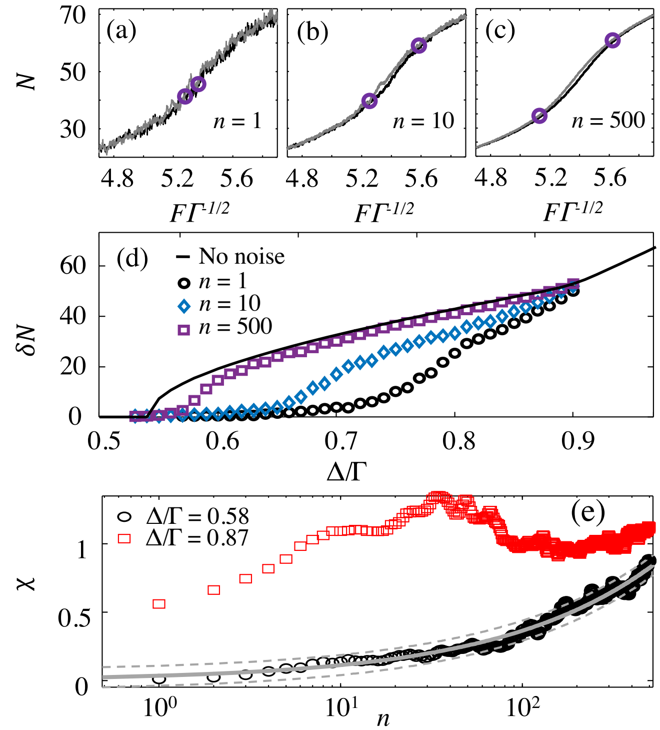

FIG. 3. (a)–(c) N versus $F/\sqrt{\Gamma}$ when scanning $F/\sqrt{\Gamma}$ within $\Gamma T = 10^4$. n is the number of cycles that are averaged. Black (gray) curves are forward (backward) trajectories. Purple circles indicate the crossing points $N_\pm$. (d) Splitting $\delta N = N_+ - N_-$, used as a signal for sensing, averaged over n cycles. For reference we show δN when $D = 0$ as a solid black curve. (e) Precision χ versus number of cycles for two different detunings. Solid gray line is a square-root fit, with dashed lines indicating 95% confidence bounds. Parameter values are as in Fig. 2 for $\Gamma T = 10^4$. Each curve is an ensemble average of (d) 12 and (e) 120 different noise realizations.

Figures 3(a)–3(c) show that, as n increases, the hysteresis widens, δN increases, and trajectories smoothen. Figure 3(d) shows δN versus Δ/Γ for the same three n. Notice the stochastic δN (open data points) approaching the deterministic δN (black solid curve) as n increases. For $n = 500$, the stochastic δN is approximately a square-root function of Δ/Γ for small Δ/Γ. This demonstrates the enhanced sensitivity at the square-root singularity in the presence of noise, albeit only after substantial averaging. Such a time-consuming averaging is of course detrimental for fast sensing. The situation appears to be familiar from conventional linear sensing, where averaging mitigates the effects of noise. However, we show next that the precision of our sensor depends nontrivially on the averaging time.

Figure 3(e) shows χ versus n for two distinct Δ/Γ. For each Δ/Γ, we simulated the dynamics of a perturbed ($\epsilon = \Gamma/100$) and an unperturbed cavity. Notice how Δ/Γ affects the dependence of χ on the averaging time. For $\Delta = 0.58\Gamma$, χ increases with the square root of time, as usual in linear sensors. In contrast, for $\Delta = 0.87\Gamma$ (close to Δ_c) χ increases abruptly for $n \lesssim 50$, *decreases* for

$50 \lesssim n \lesssim 100$, and then slowly increases for $n \gtrsim 100$. Remarkably, averaging 30 cycles leads to greater precision than averaging 500 cycles. This anomalous behavior is due to the nontrivial dependence of $d\delta N/d\Delta$ on n (see the Supplemental Material [60]) near the static square-root singularity. Thus, averaging plays a fundamentally different role in our sensor. Typically, more measurements increase the precision with which an observable is estimated. In contrast, here the precision in ϵ can be increased by restricting the number of measurements.

In summary, we introduced a nonlinear optical sensor where a square-root singularity enhances the sensitivity, precision, and information content of the signal, and the signal-to-noise ratio increases with the measurement speed. Crucially for applications, our sensing strategy involves simple monochromatic intensity measurements and no error-prone spectral fittings as in EP sensors. Since our sensor involves a single resonator, the cumbersome and slow task of tuning gain or loss is avoided. Instead, the singularity can be accessed dynamically using a commercially available amplitude modulator to modulate the laser power at the desired speed. All these advantages open up new opportunities for ultrafast and highly sensitive measurements in noisy environments. Our approach is limited to sufficiently nonlinear resonators displaying hysteresis. While optical hysteresis has been observed in many Kerr-nonlinear resonators [41,43,44,67], some of those systems operate at cryogenic temperatures where sensing applications are limited. An alternative approach could involve thermo-optical nonlinear resonators [46,49,52,68–73], easier to realize at room temperature but limited in speed by thermal dynamics. It remains to be seen whether those sensors can outperform linear sensors. Finally, our approach can be extended to other hysteretic systems, like acoustic [74] or mechanical [75,76] resonators, microelectromechanical systems [77,78], microwave circuits [79], or cavity magnon polaritons [80].

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